

Non-equilibrium phase transition in negotiation dynamics

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We introduce a model of negotiation dynamics whose aim is that of mimicking the mechanisms leading to opinion and convention formation in a population of individuals. The negotiation process, as opposed to “herding-like” or “bounded confidence” driven processes, is based on a microscopic dynamics where memory and feedback play a central role. Our model displays a non-equilibrium phase transition from an absorbing state in which all agents reach a consensus to an active stationary state characterized either by polarization or fragmentation in clusters of agents with different opinions. We show the existence of at least two different universality classes, one for the case with two possible opinions and one for the case with an unlimited number of opinions. The phase transition is studied analytically and numerically for various topologies of the agents’ interaction network. In both cases the universality classes do not seem to depend on the specific interaction topology, the only relevant feature being the total number of different opinions ever present in the system.

Statistical physics has recently proved to be a powerful framework to address issues related to the characterization of the collective social behavior of individuals, such as culture dissemination, the spreading of linguistic conventions, and the dynamics of opinion formation [1].

According to the “herding behavior” described in sociology [2], processes of opinion formation are usually modeled as simple collective dynamics in which the agents update their opinions following local majority [3] or imitation rules [4]. Starting from random initial conditions, the system self-organizes through an ordering process eventually leading to the emergence of a global consensus, in which all agents share the same opinion. In analogy with kinetic Ising models and contact processes [5], the presence of noise can induce non-equilibrium phase transitions from the *consensus* state to disordered configurations, in which more than one opinion is present. The principle of “bounded confidence” [6, 7], on the other hand, consists in enabling interactions only between agents that share already some cultural features (defined as discrete objects) [8] or with not too different opinions (in a continuous space) [6, 9]. By tuning some threshold parameter, transitions are observed concerning the number of opinions surviving in the (frozen) final state. This can be a situation of *consensus*, in which all agents share the same opinion, *polarization*, in which a finite number of groups with different opinions survive, or *fragmentation*, with a final number of opinions scaling with the system size.

In this Letter, we propose a model of opinion dynamics in which a consensus-polarization-fragmentation non-equilibrium phase transition is driven by external noise, intended as an ‘irresolute attitude’ of the agents in making decisions. The primary attribute of the model is that it is based on a negotiation process, in which memory

and feedback play a central role. Moreover, apart from the consensus state, no configuration is frozen: the stationary states with several coexisting opinions are still dynamical, in the sense that the agents are still able to evolve, in contrast to the Axelrod model [8].

Let us consider a population of N agents, each one endowed with a memory, in which an a priori undefined number of opinions can be stored. In the initial state, agents memories are empty. At each time step, an ordered pair of neighboring agents is randomly selected. This choice is consistent with the idea of *directed attachment* in the socio-psychological literature (see for instance [11]). The negotiation process is described by a local pairwise interaction rule: a) the first agent selects randomly one of its opinions (or creates a new opinion if its memory is empty) and conveys it to the second agent; b) if the memory of the latter contains such an opinion, with probability β the two agents update their memories erasing all opinions except the one involved in the interaction (*agreement*), while with probability $1 - \beta$ nothing happens; c) if the memory of the second agent does not contain the uttered opinion, it adds such an opinion to those already stored in its memory (*learning*). Note that, in the special case $\beta = 1$, the negotiation rule reduces to the Naming Game rule [12], a model used to describe the emergence of a communication system or a set of linguistic conventions in a population of individuals. In our modeling the parameter β plays roughly the same role as the *probability of acknowledged influence* in the socio-psychological literature [11]. Furthermore, as already stated for other models [13], when the system is embedded in heterogeneous topologies, different pair selection criteria influence the dynamics. In the *direct* strategy, the first agent is picked up randomly in the population, and the second agent is randomly selected

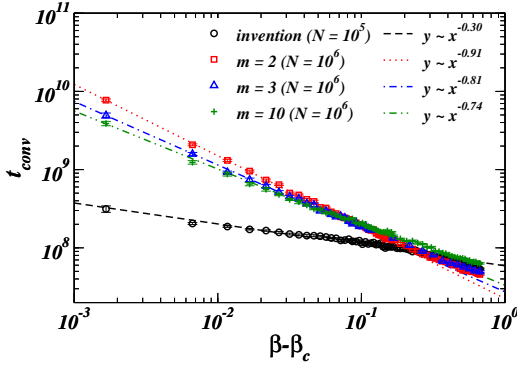


FIG. 1: (Color online) Convergence time t_{conv} of the model as a function of $\beta - \beta_c$ (with $\beta_c = 1/3$) in the case of a fully-connected population of N agents. We show data for the original model (circles) with unlimited number of opinions per agent, and for models with a finite number of opinions. Increasing m , the power-law fits give exponents that differ considerably from the value -1 .

among its neighbors. The opposite choice is called *reverse* strategy; while the *neutral* strategy consists in randomly choosing a link, assigning it an order with equal probability.

At the beginning of the dynamics, a large number of opinions is created, the total number of different opinions growing rapidly up to $\mathcal{O}(N)$. Then, if β is sufficiently large, the number of opinions decreases until only one is left and the consensus state is reached (as for the Naming Game in the case $\beta = 1$). In the opposite limit, when $\beta = 0$, opinions are never eliminated, therefore the only possible stationary state is the trivial state in which every agent possesses all opinions. Thus, a non-equilibrium phase transition is expected for some critical value β_c of the parameter β governing the update efficiency. In order to find β_c , we exploit the following general stability argument. Let us consider the consensus state, in which all agents possess the same unique opinion, say A . Its stability may be tested by considering a situation in which A and another opinion, say B , are present in the system: each agent can have either only opinion A or B , or both (AB state). The critical value β_c is provided by the threshold value at which the perturbed configuration with these three possible states does not converge back to consensus.

The simplest assumption in modeling a population of agents is the homogeneous mixing (i.e. mean-field – MF – approximation), where the behavior of the system is completely described by the following evolution equations for the densities n_i of agents with the opinion i :

$$\begin{aligned} dn_A/dt &= -n_A n_B + \beta n_{AB}^2 + \frac{3\beta-1}{2} n_A n_{AB} \\ dn_B/dt &= -n_A n_B + \beta n_{AB}^2 + \frac{3\beta-1}{2} n_B n_{AB}, \end{aligned} \quad (1)$$

and $n_{AB} = 1 - n_A - n_B$. Imposing the steady state condition $\dot{n}_A = \dot{n}_B = 0$, we get three possible solutions: 1)

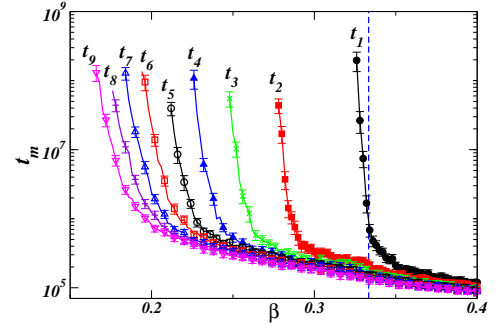


FIG. 2: (Color online) Time t_x required to a population on a fully-connected graph to reach a (fragmented) active stationary state with x different opinions. For every $m > 2$, the time t_m diverges at some critical value $\beta_c(m) < \beta_c$.

$n_A = 1, n_B = 0, n_{AB} = 0$; 2) $n_A = 0, n_B = 1, n_{AB} = 0$; and 3) $n_A = n_B = b(\beta), n_{AB} = 1 - 2b(\beta)$ with $b(\beta) = \frac{1+5\beta-\sqrt{1+10\beta+17\beta^2}}{4\beta}$ (and $b(0) = 0$). The study of the solutions' stability predicts a phase transition at $\beta_c = 1/3$. The maximum non-zero eigenvalue of the linearized system around the consensus solution becomes indeed positive for $\beta < 1/3$, i.e. the consensus becomes unstable, and the population polarizes in the $n_A = n_B$ state, with a finite density of undecided agents n_{AB} . The model therefore displays a first order non-equilibrium transition between the *frozen* absorbing consensus state and an *active* polarized state, in which global observables are stationary on average, but not frozen, i.e. the population is split in three dynamically evolving parts (with opinions A , B , and AB), whose densities fluctuate around the average values $b(\beta)$ and $1 - 2b(\beta)$.

We have checked the predictions of Eqs. (1) by numerical simulations of N agents interacting on a complete graph. Figure 1 shows that the convergence time t_{conv} required by the system to reach the consensus state indeed diverges at $\beta_c = 1/3$, with a power-law behavior $(\beta - \beta_c)^{-a}$, $a \simeq 0.3$ [18]. Very interestingly however, the analytical and numerical analysis of Eqs. (1) predicts that the relaxation time diverges instead as $(\beta - \beta_c)^{-1}$. This apparent discrepancy arises in fact because Eqs. (1) consider that the agents have at most two different opinions at the same time, while this number is unlimited in the original model (and in fact diverges with N). Numerical simulations reproducing the two opinions case allow to recover the behavior of t_{conv} predicted from Eq. (1) (see Fig. 1). We have also investigated the case of a finite number m of opinions available to the agents. The analytical result $a = 1$ holds also for $m = 3$ (but analytical analysis for larger m becomes out of reach), whereas preliminary numerical simulations performed for $m = 3, 10$ with the largest reachable population size ($N = 10^6$) lead to an exponent $a \simeq 0.74 \div 0.8$ (see Fig. 1). More extensive and systematic simulations are in order to determine the possible existence of a series of universality classes

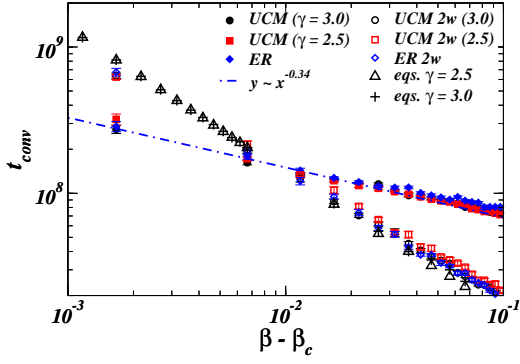


FIG. 3: (Color online) Convergence time t_{conv} of the model as a function of $\beta - \beta_c$ on networks with different topological properties: the UCM networks with degree distributions $P(k) \sim k^{-\gamma}$, $\gamma = 2.5$ and $\gamma = 3$, and the ER homogeneous random graphs. Simulations are shown for networks of $N = 10^5$ nodes and average degree $\langle k \rangle = 10$, both for $m = 2$ ("2w", open symbols) and the original model with unlimited memory (filled symbols). The numerical integration of Eqs. 2 is in good agreement with the simulations.

varying the memory size for the agents. In any case, the models with finite (m opinions) or unlimited memory define at least two clearly different universality classes for this non-equilibrium phase transition between consensus and polarized states (see [14] for similar findings in the framework of non-equilibrium q -state systems).

Figure 2 moreover shows that the transition at β_c is only the first of a series of transitions: when decreasing $\beta < \beta_c$, a system starting from empty initial conditions self-organizes into a fragmented state with an increasing number of opinions. In principle, this can be shown analytically considering the mean-field evolution equations for the partial densities when $m > 2$ opinions are present, and studying, as a function of β , the sign of the eigenvalues of a $(2^m - 1) \times (2^m - 1)$ stability matrix for the stationary state with m opinions. For increasing values of m , such a calculation becomes rapidly very demanding, thus we limit our analysis to the numerical insights of Fig. 2, from which we also get that the number of residual opinions in the fragmented state follows the exponential law $m(\beta) \propto \exp[(\beta_c - \beta)/C]$, where C is a constant depending on the initial conditions (not shown).

We now extend our analysis to more general interactions topologies, in which agents are placed on the vertices of a network, and the edges define the possible interaction patterns. When the network is a homogeneous random one (Erdős-Rényi – ER – graph [15]), the degree distribution is peaked around a typical value $\langle k \rangle$, and the evolution equations for the densities when only two opinions are present provide the same transition value $\beta_c = 1/3$ and the same exponent -1 for the divergence of t_{conv} as in MF. Figure 3 also shows that the exponent is also the MF one when the number of opinions is not limited.

Since any real negotiation process takes place on social groups, whose topology is generally far from being homogeneous, we have simulated the model on various uncorrelated heterogeneous networks (using the Uncorrelated Configuration model –UCM– model [16]), with power-law degree distributions $P(k) \sim k^{-\gamma}$ with exponents $\gamma = 2.5$ and $\gamma = 3$.

Very interestingly, the model still presents a consensus-polarization transition, in contrast with other opinion-dynamics models, like for instance the Axelrod model [17], for which the transition disappears for heterogeneous networks in the thermodynamic limit. Moreover, Fig. 3 reports the convergence time t_{conv} vs. $(\beta - \beta_c)^{-a}$, showing that at least two different universality classes are again present, one for the case with a finite ($m = 2$) number of opinions ($a = 1$) and one for the case with unlimited memory ($a \simeq 0.3$). The exponents measured are in each case compatible (up to the numerical precision) with the corresponding MF exponents (see. Fig. 3).

To understand these numerical results, we analyze, as for the fully connected case, the evolution equations for the case of two possible opinions. Such equations can be written for general correlated complex networks whose topology is completely defined by the degree distribution $P(k)$, i.e. the probability that a node has degree k , and by the degree-degree conditional probability $P(k'|k)$ that a node of degree k' is connected to a node of degree k (Markovian networks). Using partial densities $n_A^k = N_A^k/N_k$, $n_B^k = N_B^k/N_k$ and $n_{AB}^k = N_{AB}^k/N_k$, i.e. the densities on classes of degree k , one derives mean-field type equations in analogy with epidemic models. Let us consider for definiteness the neutral pair selection strategy, the equation for n_A^k is in this case

$$\begin{aligned} \frac{dn_A^k}{dt} = & -\frac{kn_A^k}{\langle k \rangle} \sum_{k'} P(k'|k) n_B^{k'} - \frac{kn_A^k}{2\langle k \rangle} \sum_{k'} P(k'|k) n_{AB}^{k'} + \\ & + \frac{3\beta kn_{AB}^k}{2\langle k \rangle} \sum_{k'} P(k'|k) n_A^{k'} + \frac{\beta kn_{AB}^k}{\langle k \rangle} \sum_{k'} P(k'|k) n_{AB}^{k'}, \quad (2) \end{aligned}$$

and similar equations hold for n_B^k and n_{AB}^k . The first term corresponds to the situation in which an agent of degree k' and opinion B chooses as second actor an agent of degree k with opinion A . The second term corresponds to the case in which an agent of degree k' with opinions A and B chooses the opinion B , interacting with an agent of degree k and opinion A . The third term is the sum of two contributions coming from the complementary interaction; while the last term accounts for the increase of agents of degree k and opinion A due to the interaction of pairs of agents with AB opinion in which the first agent chooses the opinion A .

Let us define $\Theta_i = \sum_{k'} P(k'|k) n_i^{k'}$, for $i = A, B, AB$. Under the uncorrelation hypothesis for the degrees of neighboring nodes, i.e. $P(k'|k) = k'P(k')/\langle k \rangle$, we get the

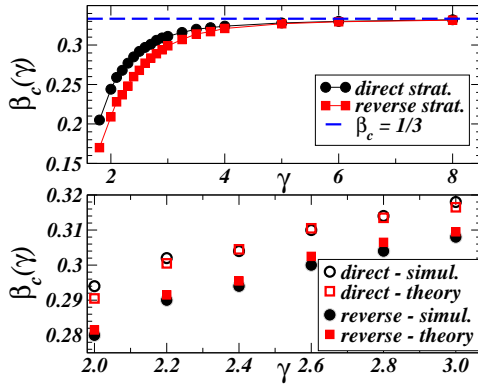


FIG. 4: (Color online) Behavior of the critical value β_c as a function of the exponent γ of the degree distribution $P(k) \sim k^{-\gamma}$, as obtained from the numerical solution of the evolution equations for n_i^k , for both direct (black circles) and reverse (red squares) strategies. Bottom: comparison between the values of $\beta_c(\gamma)$ obtained from the equations and from numerical simulations on UCM networks of $N = 10^3$ agents, for direct (open symbols) and reverse (full symbols) strategies.

following relation for the total densities $n_i = \sum_k P(k)n_i^k$,

$$\frac{d(n_A - n_B)}{dt} = \frac{3\beta - 1}{2} \Theta_{AB}(\Theta_A - \Theta_B). \quad (3)$$

If we consider a small perturbation around the consensus state $n_A = 1$, with $n_A^k \gg n_B^k$ for all k , we can argue that $\Theta_A - \Theta_B = \sum_k kP(k)(n_A^k - n_B^k)/\langle k \rangle$ is still positive, i.e. the consensus state is stable only for $\beta > 1/3$. In other words, the transition point does not change in heterogeneous topologies when the neutral strategy is assumed. This is in agreement with our numerical simulations, and in contrast with the other selection strategies. Figure 4 displays indeed the values of the critical parameter $\beta_c(\gamma)$ as a function of the exponent γ as computed from the evolution equations of the densities n_i^k (that can be derived similarly to Eqs. (2)), and as obtained from numerical simulations. In such topologies, the phase transition is shifted towards lower values of β , both for direct and reverse strategies, revealing that a preferential bias in the choice of the role played by hubs has a strong effect on the negotiation process. Reducing the skewness of $P(k)$ (increasing γ), the critical value of β converges to $1/3$.

In conclusion, we have proposed a new model of opinion dynamics based on agents negotiation in which instead memory and feedback are the essential ingredients. We have shown that a non-trivial consensus-polarization-fragmentation phase transition is observed in terms of a control parameter representing the efficiency of the negotiation process. We have elucidated the mean-field dynamics, on the fully connected graph as well as on homogeneous and heterogeneous complex networks, using a simple continuous approach. We have shown that the model presents a discontinuous phase transition between consensus and polarized states featuring at least two dif-

ferent universality classes, one for the case with $m = 2$ opinions and one for the case with an unlimited number of opinions. In both cases we have measured the critical exponent describing the divergence of the convergence time and shown that they do not seem to depend on the specific interaction topology. We argue that systems with any finite number m of opinions should fall in the $m = 2$ class. Although this point clearly deserves a deeper numerical investigation, we expect that the behavior of the model with initial invention (unlimited memory) may be due to the different spatial and temporal organization of opinions in the inventories. It would also be interesting to study the more realistic scenario in which the ‘irresolute attitude’ of the agents is modeled as a quenched disorder rather than a global external parameter. *Acknowledgments* The authors wish to thank V.D.P. Servadio for an important observation concerning eq. (1). A. Baronchelli and V. L. are partially supported by the EU under contract IST-1940 (ECAGents). A. Barrat and L.D. are partially supported by the EU under contract 001907 (DELIS).

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[18] The low value of a , which moreover slightly decreases as the system size increases, does not allow to exclude a logarithmic divergence. This issue deserves more investi-

gations that we leave for future work.